## HW 8 solutions:

**Problem 5.1 (a, b, d, e)**

(a) Let the potential in the region $-L/2 < x < L/2$ be zero and infinity elsewhere. The eigenfunctions are $\psi_n(x, t) = \sqrt{2/L}\sin(k_n(x+L/2))e^{-i\omega_n t}$, where $k_n = n\pi/L$, $\omega_n = \hbar k_n^2/2m$, and $n = 1, 2, 3, \ldots$. Hence,

$$\psi_1(x, t) = \sqrt{2/L}\sin((\pi/L)(x+L/2))e^{-i\omega_1 t} = \sqrt{2/L}\cos((\pi x/L)e^{-i\omega_1 t})$$

$$\psi_4(x, t) = \sqrt{2/L}\sin((4\pi/L)(x+L/2))e^{-i\omega_4 t} = \sqrt{2/L}\sin(4\pi x/L)e^{-i\omega_4 t}$$

and the probability distribution of the particle in the superposition state

$$\psi(x, t) = \sqrt{1/2}(\psi_1(x, t) + \psi_4(x, t))$$

is

$$|\psi(x, t)|^2 = \frac{1}{2}(|\psi_1(x)|^2 + |\psi_4(x)|^2 + 2\mathrm{Re}\{\psi_1(x)\psi_4^*(x)\}\cos((\omega_4 - \omega_1)t))$$

$$|\psi(x, t)|^2 = \frac{1}{L}\left(\cos^2\left(\frac{x\pi}{L}\right) + \sin^2\left(\frac{4\pi x}{L}\right) + 2\cos\left(\frac{\pi x}{L}\right)\sin\left(\frac{4\pi x}{L}\right)\cos\left(\frac{\hbar}{2m}\frac{15\pi^2}{L^2}t\right)\right)$$

where we note that $\omega_4 - \omega_1 = \frac{\hbar}{2m}\frac{15\pi^2}{L^2}$.

(b) Expectation value of position is

$$\langle x(t)\rangle = \int|\psi(x, t)|^2 x dx$$

$$\langle x(t)\rangle = \frac{1}{2}\int_{-L/2}^{L/2}(|\psi_1(x)|^2 x + |\psi_4(x)|^2 x + 2x\mathrm{Re}\{\psi_1(x)\psi_4^*(x)\}\cos((\omega_4 - \omega_1)t))dx$$

By symmetry, the first two terms in the integrand are zero, leaving

$$\langle x(t)\rangle = \cos((\omega_4 - \omega_1)t)\int_{-L/2}^{L/2}\frac{2}{L}\cos\left(\frac{\pi x}{L}\right)\sin\left(\frac{4\pi x}{L}\right)x dx$$

making use of $2\sin(x)\cos(y) = \sin(x+y) + \sin(x-y)$ gives

$$\langle x(t)\rangle = \cos((\omega_4 - \omega_1)t)\frac{1}{L}\int_{-L/2}^{L/2}\left(\sin\left(\frac{5\pi x}{L}\right) + \sin\left(\frac{3\pi x}{L}\right)\right)x dx$$

which, after integrating by parts, results in

$$\langle x(t)\rangle = \cos((\omega_4 - \omega_1)t)\frac{1}{L}\left(\frac{2L^2}{25\pi^2} - \frac{2L^2}{9\pi^2}\right)$$

$$\langle x(t)\rangle = -\frac{32L}{225\pi^2}\cos((\omega_4 - \omega_1)t)$$

(d) The expectation value of momentum is

$$\langle p_x(t)\rangle = -i\hbar\int\psi^*(x,t)\frac{\partial}{\partial x}\psi(x,t)\,dx = \frac{-i\hbar}{2}\int_{-L/2}^{L/2}\left((\psi_1^*+\psi_4^*)\frac{\partial}{\partial x}(\psi_1+\psi_4)\right)dx$$

The terms $\int_{-L/2}^{L/2}\psi_4^*\frac{\partial\psi_4}{\partial x}dx = 0$ and $\int_{-L/2}^{L/2}\psi_1^*\frac{\partial\psi_1}{\partial x}dx = 0$ by symmetry, leaving

$$\langle p_x(t)\rangle = \frac{-i\hbar}{2}\frac{2}{L}\frac{4\pi}{L}e^{-i(\omega_4-\omega_1)t}\int_{-L/2}^{L/2}\cos\left(\frac{\pi x}{L}\right)\cos\left(\frac{4\pi x}{L}\right)dx +$$

$$\frac{i\hbar}{2}\frac{2}{L}\frac{\pi}{L}e^{i(\omega_4-\omega_1)t}\int_{-L/2}^{L/2}\sin\left(\frac{4\pi x}{L}\right)\sin\left(\frac{\pi x}{L}\right)dx$$

$$\langle p_x(t)\rangle = \frac{-i\hbar}{2}\frac{8\pi}{L^2}e^{-i(\omega_4-\omega_1)t}\frac{2L}{15\pi} + \frac{i\hbar}{2}\frac{2\pi}{L^2}e^{i(\omega_4-\omega_1)t}\frac{8L}{15\pi}$$

$$\langle p_x(t)\rangle = \frac{-i\hbar 8}{15L}(e^{-i(\omega_4-\omega_1)t} + e^{i(\omega_4-\omega_1)t})$$

and

$$\langle p_x(t)\rangle = \frac{16\hbar}{15L}\sin((\omega_4-\omega_1)t)$$

(e) The current density is

$$J_x(x,t) = \frac{-ie\hbar}{2m}\left(\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right)$$

$$J_x(x,t) = \frac{-ie\hbar}{4m}\left((\psi_1^*+\psi_4^*)\frac{\partial}{\partial x}(\psi_1+\psi_4) - (\psi_1+\psi_4)\frac{\partial}{\partial x}(\psi_1^*+\psi_4^*)\right)$$

$$J_x(x,t) = \frac{-ie\hbar}{2Lm}\left(\cos\left(\frac{\pi x}{L}\right)\frac{\partial}{\partial x}\sin\left(\frac{4\pi x}{L}\right)(e^{i(\omega_1-\omega_4)t} - e^{-i(\omega_1-\omega_4)t}) + \right.$$

$$\left.\sin\left(\frac{4\pi x}{L}\right)\frac{\partial}{\partial x}\cos\left(\frac{\pi x}{L}\right)(e^{-i(\omega_1-\omega_4)t} - e^{i(\omega_1-\omega_4)t})\right)$$

$$J_x(x,t) = \frac{-e\hbar}{Lm}\left(\cos\left(\frac{\pi x}{L}\right)\frac{\partial}{\partial x}\sin\left(\frac{4\pi x}{L}\right) - \sin\left(\frac{4\pi x}{L}\right)\frac{\partial}{\partial x}\cos\left(\frac{\pi x}{L}\right)\right)\sin((\omega_4-\omega_1)t)$$

$$J_x(x,t) = \frac{-\pi e\hbar}{mL^2}\left(4\cos\left(\pi\frac{x}{L}\right)\cos\left(\left(4\pi\frac{x}{L}\right) + \sin\left(4\pi\frac{x}{L}\right)\sin\left(\pi\frac{x}{L}\right)\right)\right)\sin((\omega_4-\omega_1)t)$$

**Problem 5.6**

The Hamiltonian $\hat{H}$ in the Schrödinger equation is Hermitian and so it is its own Hermitian adjoint, i.e., $\hat{H} = \hat{H}^\dagger$. The time dependence of the expecation value of the observable $A$ associated with operator $\hat{A}$ is found from

$$\frac{d}{dt}\langle A\rangle = \langle\frac{\partial\psi}{\partial t}|\hat{A}|\psi\rangle + \langle\psi|\frac{\partial}{\partial t}\hat{A}|\psi\rangle + \langle\psi|\hat{A}|\frac{\partial\psi}{\partial t}\rangle$$

which may be rewritten as

$$\frac{d}{dt}\langle A\rangle = \frac{i}{\hbar}\langle\psi|\hat{H}\hat{A}|\psi\rangle - \frac{i}{\hbar}\langle\psi|\hat{A}\hat{H}|\psi\rangle + \langle\psi|\frac{\partial}{\partial t}\hat{A}|\psi\rangle$$

$$\frac{d}{dt}\langle A\rangle = \frac{i}{\hbar}\langle\psi|\hat{H}\hat{A} - \hat{A}\hat{H}|\psi\rangle + \langle\psi|\frac{\partial}{\partial t}\hat{A}|\psi\rangle$$

where we used the Hermitian property of $\hat{H}$ and the fact that the Schrödinger equation is

$$\frac{-i}{\hbar}\hat{H}|\psi\rangle = |\frac{\partial\psi}{\partial t}\rangle$$

Hence,

$$\frac{d}{dt}\langle A\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{A}]\rangle + \langle\frac{\partial}{\partial t}\hat{A}\rangle$$

If the Hamiltonian commutes with the operator $\hat{A}$, then $[\hat{H},\hat{A}] = 0$, and we may conclude that

$$\frac{d}{dt}\langle A\rangle = \langle\frac{\partial}{\partial t}\hat{A}\rangle$$

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**Problem 5.7**

(a) The position operator $\hat{x}$ is Hermitian if it is its own Hermitian adjoint, i.e., $\hat{x}^\dagger = \hat{x}$ or $\langle\psi|x|\phi\rangle = \langle\phi|x|\psi\rangle^*$. For the operator $\hat{x}$ this is easy to show

$$\langle\psi|x\psi\rangle = \int_{-\infty}^{\infty}\psi^*(x)(x\psi(x))\,dx = \int_{-\infty}^{\infty}(x\psi^*(x))\psi(x)\,dx = \int_{-\infty}^{\infty}(x\psi(x))^*\psi(x)\,dx = \langle x\psi|\psi\rangle$$

(b) To show that the operator $\frac{d}{dx}$ is anti-Hermitian we integrate by parts and make use of the fact that the wavefunction $\psi(x) \to 0$ as $x \to \pm\infty$

$$\langle\psi|\frac{d}{dx}\psi\rangle = \int_{-\infty}^{\infty} \psi^*(x)\left(i\hbar\frac{d}{dx}\psi(x)\right)dx = \psi^*(x)\psi(x)|_{x=-\infty}^{x=\infty} - \int_{-\infty}^{\infty}\left(\frac{d}{dx}\psi^*(x)\right)\psi(x)\,dx$$

$$\langle\psi|\frac{d}{dx}\psi\rangle = -\int_{-\infty}^{\infty}\left(\frac{d}{dx}\psi(x)\right)^*\psi(x)\,dx = -\langle\frac{d}{dx}\psi|\psi\rangle$$

and $\left(\frac{d}{dx}\right)^\dagger = -\frac{d}{dx}$.

(c) Because the operator $\frac{d}{dx}$ is anti-Hermitian, it follows that the momentum operator $-i\hbar\frac{d}{dx}$ is Hermitian since $\left(-i\hbar\frac{d}{dx}\right)^\dagger = i\hbar\left(-\frac{d}{dx}\right) = -i\hbar\frac{d}{dx}$.

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**Problem 5.8**

(a) The Hamiltonian is

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$$

and the momentum operator is

$$\hat{p} = -i\hbar\frac{d}{dx}$$

To find the commutator $[\hat{H}, \hat{p}]$ we have it act on the arbitrary wave function $\psi(x)$ so that

$$[\hat{H}, \hat{p}]\psi = \hat{H}\hat{p}\psi - \hat{p}\hat{H}\psi$$

$$= \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V\right)\left(-i\hbar\frac{d\psi}{dx}\right) - \left(-i\hbar\frac{d}{dx}\right)\left(-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi\right)$$

$$= i\frac{\hbar^3}{2m}\frac{d^3\psi}{dx^3} - i\hbar V\frac{d\psi}{dx} - i\frac{\hbar^3}{2m}\frac{d^3\psi}{dx^3} + i\hbar\psi\frac{dV}{dx} + i\hbar V\frac{d\psi}{dx}$$

$$= i\hbar\frac{dV}{dx}\psi$$

and

$$[\hat{H}, \hat{p}] = i\hbar\frac{dV}{dx}$$

(b) Solutions of the time-independent Schrödinger equation are eigenfunctions of the Hamiltonian. If the solutions are also eigenstates of the momentum operator then the operators $\hat{H}$ and $\hat{p}$ must commute, i.e. $[\hat{H}, \hat{p}] = 0$. Hence, from part (a), we require $[\hat{H}, \hat{p}] = i\hbar\frac{dV}{dx} = 0$. This can only be satisfied if the potential is a constant (which can be set to zero) over all space. We may conclude that only free particles can be in such eigenstates of momentum.